Birzeit University
Mathematics Department
Second Semester 2021/2022
STAT3321 – Midterm Exam
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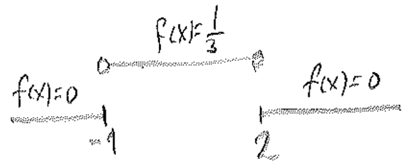
40

Name (بالعربية): ...Key............ **Student No.:** ....

Question 1 (10 points)

Let the random variable X have the following p.d.f.

$$f(x) = \begin{cases} \frac{1}{3} & -1 < x < 2 \\ 0 & elsewhere \end{cases}$$



1) Find $\Pr(-0.25 \le X < 3.75)$
2) Use the definition of the expected value to find $E(X)$.
3) Find the CDF $F(x)$.
4) Find the m.g.f. $M(t)$, if it exists.

2 1) $\Pr(-0.25 \le X < 3.75) = \int_{-0.25}^{3.75} f(x)\,dx = \int_{-0.25}^{2} \frac{1}{3}\,dx = \frac{1}{3}(2+0.25) = 0.75$

2 2) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-1}^{2} x \frac{1}{3}\,dx = \frac{1}{3}\left[\frac{x^2}{2}\Big|_{-1}^{2}\right] = \frac{1}{3}\left[\frac{3}{2}\right] = 0.50$

3 3) $F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} \int_{-\infty}^{x} 0\,dt = 0 & , x < -1 \\ \int_{-1}^{x} \frac{1}{3}\,dt = \frac{x+1}{3} & , -1 \le x < 2 \\ \int_{-1}^{2} \frac{1}{3}\,dt = 1 & , 2 \le x \end{cases}$

4) $E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-1}^{2} e^{tx}\frac{1}{3}\,dx \overset{t\neq 0}{=} \left[\frac{e^{tx}}{3t}\Big|_{-1}^{2}\right] = \frac{e^{2t}-e^{-t}}{3t}$

3 $M(t) = \begin{cases} \frac{e^{2t}-e^{-t}}{3t} & , t \neq 0 \\ 1 & , t = 0 \end{cases}$

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Question 2 (10 points)

Let the random variable X have the following p.d.f.

$$f(x) = \begin{cases} 3\left(\frac{1}{4}\right)^x & x = 1,2,3,\dots \\ 0 & elsewhere \end{cases}$$

1) Find $\Pr(-5 \le X < 2.5)$
2) Find the m.g.f. $M(t)$, if it exists.
3) Find $E(X)$.

1) $\Pr(-5 \le X < 2.5) = \sum_{x \in B} f(x)$, $B = \{x : -5 \le x < 2.5\}$

4 $= f(1) + f(2) = 3\left(\frac{1}{4}\right)^1 + 3\left(\frac{1}{4}\right)^2$

$= \frac{3}{4} + \frac{3}{16} = \frac{15}{16} = 0.9375$

2) $E(e^{tX}) = \sum_{x \in \mathbb{R}} e^{tx} f(x) = \sum_{x=1}^{\infty} e^{tx}\, 3\left(\frac{1}{4}\right)^x$

$= 3 \sum_{x=1}^{\infty} \left(\frac{e^t}{4}\right)^x = 3 \frac{e^t/4}{1 - e^t/4}$ if $\left|\frac{e^t}{4}\right| < 1$

4 $= \frac{3e^t}{4 - e^t}$ if $t < \ln 4$

$M(t) = \frac{3e^t}{4 - e^t}$, $t < \ln 4$

3) $M'(t) = \frac{(3e^t)(4 - e^t) - (-e^t)(3e^t)}{(4 - e^t)^2}$

$M'(0) = \frac{(3)(4-1) - (-1)(3)}{(4-1)^2} = \frac{9+3}{9} = \frac{12}{9} = \frac{4}{3}$

2 $E(X) = \frac{4}{3} = 1.33$

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Question 3 (10 points)

Let the random variables X and Y have the joint p.d.f.

(x,y)	(1,1)	(2,1)	(4,1)	(1,3)	(2,3)	(4,3)
$f(x,y)$	$\frac{1}{20}$	$\frac{6}{20}$	$\frac{5}{20}$	$\frac{3}{20}$	$\frac{2}{20}$	$\frac{3}{20}$

1) Find $E(XY)$.
2) Find $f_2(y)$ the marginal p.d.f. of Y.
3) Find $f_{1|2}(x|1)$ the conditional p.d.f. of X given $Y = 1$.
4) Find $E(X|Y = 1)$ the conditional expected value of X given $Y = 1$.

1) $E(XY) = \sum_{y\in\mathbb{R}}\sum_{x\in\mathbb{R}} xy\, f(x,y)$

$= (1)(1)\left(\frac{1}{20}\right) + (2)(1)\left(\frac{6}{20}\right) + (4)(1)\left(\frac{5}{20}\right) +$

$(1)(3)\left(\frac{3}{20}\right) + (2)(3)\left(\frac{2}{20}\right) + (4)(3)\left(\frac{3}{20}\right) = \frac{90}{20} = 4.5$

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2) $f_2(y) = \sum_{x\in\mathbb{R}} f(x,y) = \begin{cases} \sum_{x\in\mathbb{R}} f(x,1) = \frac{1}{20} + \frac{6}{20} + \frac{5}{20} = \frac{12}{20}, & y = 1 \\ \sum_{x\in\mathbb{R}} f(x,3) = \frac{3}{20} + \frac{2}{20} + \frac{3}{20} = \frac{8}{20}, & y = 3 \\ 0, & \text{else} \end{cases}$

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3) $f_{1|2}(x|1) = \frac{f(x,1)}{f_2(1)} = \frac{f(x,1)}{\frac{12}{20}} = \begin{cases} \frac{1/20}{12/20} = \frac{1}{12}, & x = 1 \\ \frac{6/20}{12/20} = \frac{6}{12}, & x = 2 \\ \frac{5/20}{12/20} = \frac{5}{12}, & x = 4 \\ 0, & \text{else} \end{cases}$

3

4) $E(X|Y=1) = \sum_{x\in\mathbb{R}} x\, f_{1|2}(x|1) = (1)\left(\frac{1}{12}\right) + (2)\left(\frac{6}{12}\right) + (4)\left(\frac{5}{12}\right)$

$= \frac{33}{12} = 2.75$

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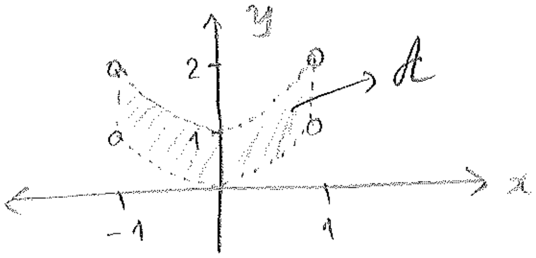
Question 4 (10 points)

Let the random variables X and Y have the following the joint p.d.f.

$$f(x,y) = \begin{cases} K, & x^2 < y < x^2+1, \quad -1 < x < 1, \\ 0, & otherwise \end{cases}$$

1) Draw $\mathcal{A}$.
2) Find the value of the constant K.
3) Are the random variables X and Y independent? Explain.
4) Find $E(Y)$.

2 1)



3 2) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{-1}^{1}\int_{x^2}^{x^2+1} K\,dy\,dx = K\int_{-1}^{1}(x^2+1-x^2)\,dx$

$= K\int_{-1}^{1} dx = K(2) = 2K$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow 2K = 1 \Rightarrow K = \frac{1}{2}$

2 3) X and Y are <u>not</u> independent because $f(x,y) \neq h(x)\cdot g(y)$ for all $(x,y) \in \mathbb{R}^2$

3 4) $E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\,dx\,dy = \int_{-1}^{1}\int_{x^2}^{x^2+1} y\,K\,dy\,dx$

$= K\int_{-1}^{1}\left[\int_{x^2}^{x^2+1} y\,dy\right]dx = K\int_{-1}^{1}\left[\frac{y^2}{2}\Big|_{x^2}^{x^2+1}\right]dx$

$= \frac{K}{2}\int_{-1}^{1}\left[(x^2+1)^2 - (x^2)^2\right]dx = \frac{K}{2}\int_{-1}^{1}(x^4+2x^2+1-x^4)\,dx$

$= \frac{K}{2}\int_{-1}^{1}(2x^2+1)\,dx = \frac{K}{2}\left[\left(\frac{2x^3}{3}+x\right)\Big|_{-1}^{1}\right] = \frac{K}{2}\left[\frac{4}{3}+2\right]$

$= \frac{K}{2}\left[\frac{10}{3}\right] = \frac{10K}{6} = \frac{10}{12} = \frac{5}{6} = 0.83$

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